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THE ASYMPTOTIC SOLUTION OF A CONTACT PROBLEM WITH A HALF-UNKNOWN BOUNDARY OF THE CONTACT REGION†

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The following problem is considered: a punch, disk-shaped in plan, whose base is an almost circular elliptical paraboloid, is pressed onto the boundary of an elastic half-space. The friction between the bodies in contact is ignored; the punch edges are allowed to separate from the elastic base. It is assumed that the contact region, which is not known a priori, is almost circular. The solution of the problem is constructed by the method of matched asymptotic expansions. The asymptotic form of the boundary of the contact region is presented in explicit form.

1. FORMULATION OF THE PROBLEM

Suppose a punch whose base is an elliptical paraboloid

$$x_3 = -\Phi_\varepsilon(x_1, x_2); \quad \Phi_\varepsilon(x_1, x_2) = A[x_1^2 + x_2^2 + \varepsilon(x_1^2 - x_2^2)] \tag{1.1}$$

where ε is a small positive parameter, is pressed without friction into the elastic half-space $x_3 > 0$. It is assumed that in plan the punch occupies a disk ω_0 of radius a with centre at the origin.

Letting δ_ε denote the forward displacement of the punch, we set

$$\delta_\varepsilon = \delta_0 + \varepsilon\delta_1; \quad \delta_0 = 2Aa^2 \tag{1.2}$$

Then (see, e.g. [1], Chap. 3, Sec. 2) when $\varepsilon = 0$ the punch is in contact with the base over the entire region ω_0, and the contact pressure vanishes at the boundary Γ_0.

In the general situation, the contact region ω_ε turns out to be some subregion of ω_0 which is not known a priori. Using the Papkovich–Neuber representation, one can reduce the problem [2, 3] to that of determining a harmonic function u^ε that vanishes at infinity and satisfies the following boundary conditions

$$\begin{aligned} &u^\varepsilon(\mathbf{x}) \geqslant \delta_\varepsilon - \Phi_\varepsilon(x_1, x_2), \quad \partial_3 u^\varepsilon(\mathbf{x}) \leqslant 0 \\ &[u^\varepsilon(\mathbf{x}) - \delta_\varepsilon + \Phi_\varepsilon(x_1, x_2)]\partial_3 u^\varepsilon(\mathbf{x}) = 0 \quad x_3 = 0, \ (x_1, x_2) \in \omega_0 \end{aligned} \tag{1.3}$$

$$\partial_3 u^\varepsilon(\mathbf{x}) = 0, \quad x_3 = 0, \quad (x_1, x_2) \in \mathbf{R}^2 \setminus \overline{\omega}_0 \tag{1.4}$$

The boundary Γ_ε of the contact region ω_ε is defined by the condition that the contact pressures

$$p^\varepsilon(x_1, x_2) = -\alpha^{-1}\partial_3 u^\varepsilon(x_1, x_2, 0); \quad \alpha \equiv 2(1 - \nu^2)E^{-1} \tag{1.5}$$

must be positive, where E and ν are Young's modulus and Poisson's ratio of the elastic half-space.

In a previously obtained analytical solution of the problem [4], the coefficients of the trigonometric expansion of a function h_ε, characterizing the deviation of the contact region from the disk, were computed by the collocation method. In this paper an asymptotic method developed by Nazarov [5, 6] will be employed. A simple closed expression will be derived for a piecewise-smooth function h_ε.

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2. THE OUTER ASYMPTOTIC EXPANSION

Far from the disturbed part of the contact boundary, we prescribe the solution of problem (1.3), (1.4) in the form

$$u^{\varepsilon}(\mathbf{x}) = v^0(\mathbf{x}) + \varepsilon v^1(\mathbf{x}) \tag{2.1}$$

where the right-hand side is the solution of the linear contact problem of the punch (1.1) pressed in to the depth (1.2). Accordingly, we have the following representation for the contact pressure

$$p^{\varepsilon}(x_1, x_2) = p^0(x_1, x_2) + \varepsilon p^1(x_1, x_2) \tag{2.2}$$

Using well known results (see [1, 2, 7], etc.), we find

$$p^0(x_1, x_2) = \frac{E}{\pi(1-\nu^2)} \frac{2\delta_0}{a^2} \sqrt{a^2 - x_1^2 - x_2^2} \tag{2.3}$$

$$p^1(x_1, x_2) = \frac{E}{\pi(1-\nu^2)} \frac{\delta_1 - (8/3)A(x_1^2 - x_2^2)}{\sqrt{a^2 - x_1^2 - x_2^2}} \tag{2.4}$$

Let us determine the behaviour of the function (2.1) in the neighbourhood of the contact boundary. In a three-dimensional neighbourhood of the contour Γ_0, we introduce local coordinates $y_1, y_2 = x_3, \sigma$, where σ is the polar angle. In addition, in planes orthogonal to Γ_0 we introduce polar coordinates r and φ, where $y_1 = r\cos\varphi$, $y_2 = r\sin\varphi$ and $\varphi \in [0, \pi]$.

Direct computations for the densities (2.3) and (2.4) yield the following asymptotic formulae

$$p^0(x_1, x_2) = -(2\pi)^{-1/2} k^0(\sigma) r^{1/2} + O(r^{3/2}), \quad r \to 0 \tag{2.5}$$

$$p^1(x_1, x_2) = -(2\pi)^{-1/2} K^1(\sigma) r^{-1/2} + O(r^{1/2}), \quad r \to 0 \tag{2.6}$$

where, by normalization as in [5], we have

$$k^0(\sigma) = -\frac{E}{1-\nu^2} \frac{4\delta_0}{\sqrt{\pi} a^{3/2}} \tag{2.7}$$

$$K^1(\sigma) = -\frac{E}{1-\nu^2} \left(\frac{\delta_1}{\sqrt{\pi a}} - \frac{8Aa^2}{3\sqrt{\pi a}} \cos 2\sigma \right) \tag{2.8}$$

Finally, for the functions v^0 and v^1 we have the following asymptotic expansions as $r \to 0$ (see for example [8, 5]).

$$v^0(\mathbf{x}) = Aa^2 + 2Aar\cos\varphi + \alpha(2\pi^{-1})^{1/2} 3^{-1} k^0(\sigma) r^{3/2} \sin(3\varphi/2) + O(r^2) \tag{2.9}$$

$$v^1(\mathbf{x}) = \delta_1 - Aa^2 \cos 2\sigma + \alpha(2\pi^{-1})^{1/2} K^1(\sigma) r^{1/2} \sin(\varphi/2) + O(r) \tag{2.10}$$

Note that only the second inequality is violated for the function (2.3) in boundary condition (1.3), in the neighbourhood of those points of the contour Γ_0 where the stress intensity factor (2.8) is positive.

3. THE INNER ASYMPTOTIC EXPANSION

In planes normal to Γ_0 we change to "stretched" variables

$$\eta = (\eta_1, \eta_2); \quad \eta_i = \varepsilon^{-1} y_i \tag{3.1}$$

The solution of problem (1.3), (1.4) near Γ_0 will be sought in the form

$$u^{\varepsilon}(\mathbf{x}) \simeq w^0(\sigma;\boldsymbol{\eta})+\varepsilon^{1/2}w^1(\sigma;\boldsymbol{\eta})+\varepsilon w^2(\sigma;\boldsymbol{\eta})+\varepsilon^{3/2}w^3(\sigma;\boldsymbol{\eta}) \tag{3.2}$$

We substitute expansions (2.9) and (2.10) into (2.1) and apply the coordinate transformation inverse to (3.1). Setting $r = \varepsilon\rho$, we obtain

$$v^0(\mathbf{x})+\varepsilon v^1(\mathbf{x}) = Aa^2+\varepsilon[2Aa\eta_1+\delta_1-Aa^2\cos 2\sigma]+$$

$$+\varepsilon^{3/2}\alpha(2\pi^{-1})^{1/2}[3^{-1}k^0(\sigma)\rho^{3/2}\sin(3\varphi/2)+K^1(\sigma)\rho^{1/2}\sin(\varphi/2)]+O(\varepsilon^2\rho^2) \tag{3.3}$$

By the method of matched asymptotic expansions (see [9–11] and others), the first terms on the right in (3.3) are determined by the leading terms of the boundary layer

$$w^0(\sigma;\boldsymbol{\eta})\equiv Aa^2,\quad w^1(\sigma;\boldsymbol{\eta})\equiv 0,\quad w^2(\sigma;\boldsymbol{\eta})\equiv 2Aa\eta_1+\delta_1-Aa^2\cos 2\sigma \tag{3.4}$$

By the second equality in (3.4), the function w^3 must be harmonic in the half-plane $\eta_2 > 0$ (see [5, Section 3])

4. VARIATION OF THE BOUNDARY OF THE CONTACT REGION

Let us assume that the curve Γ_ε is described in local coordinates by the equation

$$y_1 = h_\varepsilon(\sigma);\qquad h_\varepsilon(\sigma)=\varepsilon h(\sigma) \tag{4.1}$$

or $\eta_1 = h(\sigma)$, where $h(\sigma)$ is a function to be determined. Besides the coordinates ρ and φ, we also introduce polar coordinates ρ_h and $\varphi_h \in [0, \pi]$ with pole at the point $\eta_1 = h(\sigma)$, $\eta_2 = 0$.

Proceeding as in [6], we prescribe the lower-order term of the inner asymptotic expansion (3.2) in the form

$$w^3(\sigma;\boldsymbol{\eta})=\alpha(2\pi^{-1})^{1/2}3^{-1}N(\sigma)\rho_h^{3/2}\sin(3\varphi_h/2) \tag{4.2}$$

The function (3.2) essentially satisfies the second inequality of (1.3) in the contact region $\eta_1 > h(\sigma)$, provided that $N(\sigma) < 0$. When that happens the contact pressure will vanish over the disturbed part of the contour Γ_ε. In addition, because of (3.4) and (4.2), the first inequality of (1.3) will be an equality apart from quantities $O(\varepsilon^2)$. Finally, the functions N and h are determined by comparing the asymptotic expansions (2.1) and (3.2).

Using the formulae [5]

$$\partial r_h/\partial h = -\cos\varphi,\quad \partial\varphi_h/\partial h = r^{-1}\sin\varphi \quad \text{for}\quad h=0$$

for large values of ρ we derive the relation

$$\rho_h^{3/2}\sin(3\varphi_h/2)=\rho^{3/2}\sin(3\varphi/2)-(3/2)\rho^{1/2}h(\sigma)\sin(\varphi/2)+O(\rho^{-1/2}) \tag{4.3}$$

Substituting (4.1) into (4.2) and making the change of variables (3.1), we find that

$$\varepsilon^{3/2}w^3(\sigma;\boldsymbol{\eta})=\alpha(2\pi^{-1})^{1/2}3^{-1}N(\sigma)r^{3/2}\sin(3\varphi/2)-$$

$$-\varepsilon\alpha(2\pi^{-1})^{1/2}2^{-1}N(\sigma)h(\sigma)r^{1/2}\sin(\varphi/2)+O(\varepsilon^2 r^{-1/2}) \tag{4.4}$$

Now, comparing relations (3.3) and (3.2) taking into account equalities (3.4) and (4.4), we conclude that in the matching zone, where $r/a = O(\varepsilon^{2/3})$, the inner and outer expansions differ by quantities $O(\varepsilon^{4/3})$, provided that $N(\sigma) = k^0(\sigma)$ and

$$h(\sigma)=(-2K^1(\sigma)/k^0(\sigma))_+ \tag{4.5}$$

The subscript plus denotes the operation of taking the positive value of the expression in the parentheses. We emphasize that at those points σ where $K^1(\sigma) < 0$, so that compressive stresses develop near the edge of the punch, formula (4.5) gives $h(\sigma) = 0$ and the boundary Γ_ε coincides with Γ_0. By (2.7) and (2.8), the function (4.5) may be written in the form

$$h(\sigma) = ((4/3)a\cos 2\sigma - (1/4)\delta_1)_+$$

or, if (1.2) and (1.4) are taken into consideration

$$h_\varepsilon(\sigma) = \left(-\frac{1}{4Aa}(\delta_\varepsilon - \delta_0) + \frac{2}{3}\varepsilon a\cos 2\sigma\right)_+ \tag{4.6}$$

Formula (4.6) in the main determines the position of the boundary Γ_ε.

5. THE CONDITIONS FOR WHICH THE PUNCH EDGE DOES OR DOES NOT SEPARATE FROM THE BASE SURFACE

It is not hard to see that the contact pressure (2.2) on ω_0 will be positive if $\delta_1 - (8/3)Aa^2\cos 2\sigma \geqslant 0$ for $\sigma \in [0, 2\pi]$ or $\delta_1 - (8/3)Aa^2$. Consequently, the condition for full contact is as follows (see also [12], Sec. 1.5.2):

$$\delta_0 + \varepsilon(8/3)Aa^2 \leqslant \delta_\varepsilon \tag{5.1}$$

Note that the boundary (5.1) coincides exactly with that obtained from asymptotic formula (4.6). Indeed, if inequality (5.1) is satisfied, the function (4.6) will vanish identically.

On the other hand, formula (4.6) predicts that the whole edge of the punch will separate from the elastic base if

$$\delta_\varepsilon \leqslant \delta_0 - \varepsilon(8/3)Aa^2 \tag{5.2}$$

Let us compare this result with the exact result obtained using Hertz's formulae ([13], Chap. V, Sec. 6.5). In this case, the contact surface is bounded by an ellipse with semi-major axes a and eccentricity e defined by the equation

$$\frac{(1-e^2)[\mathbf{K}(e)-\mathbf{E}(e)]}{\mathbf{E}(e)-(1-e^2)\mathbf{K}(e)} = \frac{1-\varepsilon}{1+\varepsilon}$$

Using the expansions of the complete elliptic integrals $\mathbf{K}$ and $\mathbf{E}$ for small values of the modulus (see, e.g. [14]), we see that as $\varepsilon \to 0$

$$e^2 = (8/3)\varepsilon + O(\varepsilon^2).$$

Using the relation and the equation

$$a = \sqrt{\frac{\delta_\varepsilon}{2A}}\left(\frac{\mathbf{E}(e)}{\mathbf{K}(e)(1-e^2)}\right)^{1/2}$$

we compute the asymptotic behaviour of the displacement of the punch

$$\delta_\varepsilon = 2Aa^2 - \varepsilon(8/3)Aa^2 + O(\varepsilon^2) \tag{5.3}$$

If (1.2) is taken into consideration, formula (5.3) is in general agreement with estimate (5.2).

6. CONCLUSION

In the neighbourhood of the disturbed part of the contact boundary, one has the phenomenon of a plane boundary layer. Here the following formula holds for the contact pressure (by relations (3.2) and (4.2))

$$p^\varepsilon(x_1, x_2) = -(2\pi)^{-1/2}k^0(\sigma)\sqrt{y_1 - h_\varepsilon(\sigma)}$$

where $y_1 \geqslant h_\varepsilon(\sigma)$

It should be noted that the essential point in the asymptotic solution of the problem is the fact that if $\varepsilon = 0$, the contact pressure vanishes along the entire boundary Γ_0. In the problem of separation from an elastic base for the edge of a punch with a two-dimensional base (see [15, 16]), more complicated constructions are needed. We also note that in neighbourhoods of points where the base surface separates from the punch edge, asymptotic formula (4.5), obviously, does not described the local behaviour of the boundary Γ_ε.

In the problem considered, the punch surface is a perturbation of a circular paraboloid by an elliptic paraboloid. Using the method described in [5], an analogous study may be made of the case in which the perturbed surface is, say, of fourth order.

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